AstroChallenge 2018
Data Response Questions
(Senior)

<u>PLEASE READ THESE INSTRUCTIONS CAREFULLY</u>

1. This paper consists of **<u>21</u>** printed pages and **<u>4</u>** blank pages, including this cover page.
2. Do **NOT** turn over this page until instructed to do so.
3. You have **2 hours** to attempt all questions in this paper.
4. At the end of the paper, submit this booklet together with your answer script.
5. Your answer script should clearly indicate your school (and team number) on **EVERY** page, as well as the individuals in the said team on the first page.
6. It is your team's responsibility to ensure that all pages of your answer script have been submitted, **including pages to be detached from this booklet**.

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Question 1
Getting Crabby
(20 points)

Born from a supernova in 1054, the Crab Nebula has the distinction of being the only supernova remnant on the Messier Catalogue. Today, the Crab Nebula is known to be one of the harder winter Messier objects to observe: small scopes typically only reveal a dim glow even in good skies. Yet, plainly it was bright enough in the past for Charles Messier to mistake as a comet despite his mediocre telescope (by modern standards). This leads us to the question: how much easier was the Crab Nebula to observe in the past? Let us rewind the clock...

The table below summarises some of the current known information about the Crab pulsar.

Characteristic	**Symbol**	**Value**
Apparent magnitude	$m(V)$	$+16.5$
Distance	d	$1.9\,\text{kpc}$
Mass	M	$1.4M_\odot$
Radius	R	$10\,\text{km}$
Rotational period	P	$33\,\text{ms}$
Temperature	T	$1.6\times10^6\,\text{K}$

Table 1: Some characteristics of the Crab pulsar.

(i) [1 point] What is the current luminosity of the Crab pulsar solely due to black-body radiation L_B?

Solution.

$$\begin{aligned} L_B &= 4\pi R^2\sigma T^4 \\ &= 4\pi(10\,\text{km})^2\sigma(1.6\times10^6\,\text{K}) \\ &= 4.67\times10^{26}\,\text{W}\,. \end{aligned}$$

For comparison, the entire Crab Nebula has been determined to have a current luminosity L of 5×10^{31} W (Kennel and Coroniti, 1984). The Crab Nebula would have quickly cooled to background temperatures if the Crab Pulsar's black-body radiation was the sole source of thermal energy.

An alternative source of energy is rotational magnetic braking. We know that pulsars are rapidly rotating and have strong magnetic fields. As magnetic field lines move past the surrounding material, the surrounding material exerts a force that opposes this motion (by Lenz's law). The net result is that the pulsar loses rotational kinetic energy to its surroundings, in a process known as spin-down.

Suppose we have an object spinning with period P. It is known that the rotational kinetic energy of an object is related to its angular velocity ω by the relation

$$E_{\mathrm{rot}} = \frac{1}{2} I \omega^2,$$

where I is the moment of inertia. For the case of a perfectly uniform sphere, $I = \frac{2}{5} M R^2$, where M and R are the mass and radius of the sphere respectively.

(ii) [2 points] Assume that the Crab Pulsar is perfectly spherical with a constant moment of inertia. Show that the amount of energy emitted per second due to magnetic braking L_{spin} is given by

$$L_{\text{spin}} = 4\pi^2 I P^{-3} \frac{dP}{dt},$$

where $\frac{dP}{dt}$ is the change in the pulsar's period as the pulsar ages.

(Hint: Think of the physical picture. As the pulsar slows, the rotational kinetic energy of the pulsar is being lost to space, which is observed as L_{spin}. This implies that $L_{\text{spin}} = -\frac{d}{dt}[E_{\text{rot}}]$.)

Solution. We have

$$E_{\text{rot}} = \frac{1}{2} I\omega^2 = \frac{1}{2} I \left(\frac{2\pi}{P}\right)^2 = 2\pi^2 I P^{-2}.$$

When the pulsar decelerates, its moment of inertia remains unchanged (aka mass/radius are roughly constant). This leaves us with a simple differentiation problem, i.e.

$$L_{\text{spin}} = -\frac{d}{dt}[2\pi^2 I P^{-2}] = 4\pi^2 I P^{-3} \frac{dP}{dt}.$$

(iii) For the case of the Crab Pulsar, $\frac{dP}{dt} = 4.22 \times 10^{-13}$ (seconds per second).

(a) [1 point] Hence or otherwise, calculate L_{spin}.

(b) [1 point] By comparing this value to L_B and L, suggest if rotational magnetic braking is sufficient to power the Crab Nebula.

Solution. Plugging in values,

$$L_{\text{spin}} = 4\pi^2 \left(\frac{2}{5} \times 1.4 M_\odot \times (10\,\text{km})^2\right) (33\,\text{ms})^{-3} (4.22 \times 10^{-13}) = 5.16 \times 10^{31}\,\text{W}\,.$$

Therefore,

$$L \approx L_{\text{spin}} \gg L_B.$$

Thus, we conclude that rotational magnetic braking is the dominant source of energy for the Crab Nebula, and is sufficient.

Under suitable assumptions, the period of a pulsar and its spin-down rate are related by the formula

$$\tau = \frac{P}{2 \times \frac{dP}{dt}},$$

where τ is the characteristic age of the pulsar.

(iv) [3 points] Assuming that the characteristic age of the Crab pulsar is equal to its age t (i.e.

$\tau = t$), prove that the relationship between P and t is given by

$$P = K\sqrt{t},$$

where K is a constant. You **do not** need to compute a value for K. (its numerical value does not need to be found). [3 marks]

Solution. We have $\frac{dP}{dt} = \frac{P}{2t}$. Integrating both sides w.r.t. t, we have

$$\int \frac{1}{P}\,dP = \int \frac{1}{2t}\,dt.$$

Then,

$$\ln P = 0.5\ln t + C,$$

for some constant C. Therefore we have

$$P = e^{0.5\ln t + C} = e^C e^{\ln\sqrt{t}} = K\sqrt{t},$$

where we have written $K = e^C$. This completes the proof.

We are interested in seeing how the luminosity of the nebula changes with time, so let us figure that out next.

(v) [1 point] Let the luminosity of the nebula at time t be L_t. For simplicity, let us assume for the rest of this question that $L_t = L_{\text{spin}}$.

Express L_t as a function of t, I, K, and other numerical constants only.

Solution. We have

$$L_T = 4\pi^2 I P^{-3}\frac{dP}{dt} = 4\pi^2 I (K\sqrt{t})^{-3}\frac{K}{2\sqrt{t}} = \frac{2\pi^2 I}{K^2 t^2}.$$

(vi) [1 point] Charles Messier first observed the Crab Nebula in 1758. What is the luminosity of the Crab Nebula now compared to then? Express your answer as a percentage.

Solution. In 1758, the Crab was 704 years old. In 2018, it is now 964 years old. Hence,

$$L_t \propto \frac{1}{t^{1.5}} \qquad \Longrightarrow \qquad \frac{L_{964}}{L_{704}} = \left(\frac{704}{964}\right)^2 \approx 53.3\%.$$

(vii) [2 points] Let your numerical answer in Part (vi) be Z. Does this mean that the nebula right now (as a whole) is visually Z times as bright as it was in 1758? Why or why not?

Solution. No. Recall that luminosity considers the emission of the nebula across all wavelengths. It is perfectly possible (indeed likely) that the emission profile of the nebula has changed over time, such that it now emits relatively more radiation in certain types of light (e.g. infrared) as compared to others.

This is all well and good, but as seasoned astronomers, you should be familiar that total brightness is not worth much. Rather, in order to gauge visibility, what matters more is surface brightness. To this end, let us now attempt to find out how the average surface brightness of the Crab has changed.

The Crab Nebula itself currently has a radius of 5.5 light years and is expanding at $1500\,\mathrm{km\,s^{-1}}$, and we know the expansion rate is being slowed by gravity. We also know that the entire nebula has a mass of 4.6 solar masses, separate from the pulsar itself. While the nebula is an extended object, given that it is spherically symmetric, we can apply the Shell Theorem. The Shell Theorem allows us to treat the nebula as *decelerating due to a massive "point mass" of mass M located at its centre.* In this case, M refers to the combined mass of the nebula and the central pulsar.

(viii) [1 point] Let the radius of the nebula be R. Consider a test mass expanding with the edge of the nebula (a.k.a. R). Write down the second-order differential equation relating acceleration $\frac{d^2R}{dt^2}$ to M, R, and other constants where appropriate.

(Hint: Analyse the forces involved.)

Solution. Since gravity is the only force,

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2}.$$

(xi) [3 points] Find a relationship between the expansion velocity v and R, including any other appropriate constants. Label and explain any constants that you introduce.

(Hint: While it may be tempting to solve the differential equation in Part (viii) to obtain an answer, there is another (and better) way to approach this question.)

Solution. NB: The existing literature all assume a constant expansion rate. This problem shows why astronomers do so: a model with gravitational forces only is very tedious to solve. Furthermore, we can show that this approximation incurs negligible error in this instance. The proof of this is given at the end of the solution.

We can either deal with this problem by solving the DE, or energy conservation. As foreshadowed by the hint, you'll find that energy conservation is far easier to work with.

Method 1: From energy conservation, KE + GPE = TE. Hence,

$$\mathrm{TE} = \frac{1}{2}mv^2 - \frac{GMm}{R}.$$

Note that TE is the total energy and is constant (we cannot assume TE = 0). Dividing by m,

$$\frac{\mathrm{TE}}{m} = \frac{1}{2}v^2 - \frac{GM}{R}.$$

Simplifying,

$$v = \sqrt{\frac{2\times\mathrm{TE}}{m} + \frac{2GM}{R}}.$$

Method 2: We solve the DE. Recall that by chain rule,

$$a := \frac{dv}{dt} = \frac{dv}{dR} \cdot \frac{dR}{dt} = \frac{dv}{dR} v.$$

We now compute the integral $\int a\, dR$ in two ways. We have

$$\int a\, dR = \int -\frac{GM}{R^2}\, dR = \frac{GM}{R} + A,$$

where A is a constant. Alternatively,

$$\int a\, dR = \int \frac{dv}{dR} v\, dR = \int v\, dv = \frac{v^2}{2} + B,$$

where B is again a constant. Therefore,

$$\frac{GM}{R} = \frac{v^2}{2} + C,$$

where C is a constant, so that

$$v = \sqrt{C + \frac{2GM}{R}}.$$

This is the same equation as above with $C = \frac{2 \times \mathrm{TE}}{m}$, which should not be too surprising. Take note that the constant of integration C has a physical interpretation – it is directly related to the total energy of the system. This would not have been obvious if we had not used the energy conservation approach first.

Proof of approximately constant expansion rate (optional): Given the given values of M, R, and v presently, we can substitute in the values to find that $C = 2.25 \times 10^{12}\,\mathrm{m}^2\,\mathrm{s}^2$.

Then,

$$\frac{dR}{dt} = v = \sqrt{C + \frac{2GM}{R}},$$

so that

$$\int \frac{1}{\sqrt{C + \frac{2GM}{R}}}\, dR = \int dt.$$

Rewriting the LHS yields

$$\int \frac{\sqrt{R}}{\sqrt{RC + 2GM}}\, dR = \int dt.$$

This is not a nice integral, and we will not obtain a "clean" explicit solution for R. (Note that it can be integrated – one starts with setting $z = \sqrt{R}$, then following through with a series of substitutions.) We instead find the first-order Taylor series approximation of v with respect to the present day. This is permissible since for the purposes of the question (comparing what Messier saw to what we see now), we are not moving too far away from the present day. We then have:

$$\begin{aligned}\frac{dv}{dR} &= -\frac{GM}{R^2\sqrt{C+\frac{2GM}{R}}} \\ &\approx 1.5\times10^{6} - 1.961\times10^{-19}(R-5.2\times10^{16}) \\ &\approx 1.5\times10^{6} - 1.961\times10^{-19}R.\end{aligned}$$

This Taylor series tells us why astronomers typically assume a constant expansion velocity: 1.961×10^{-19} is extraordinarily small. For the values we're considering, $1.961\times10^{-19}R$ has order of millimetres per second, which is tiny compared to the $1500\,\mathrm{km\,s^{-1}}$ expansion rate. Indeed, despite the 260 year interval involved in our question, the error involved in this approximation is on the order of 100000 km. This is negligible when we recall the current radius of the nebula (5.5 light years)!

It can be shown that if we plug in the given values and adopt certain simplifying assumptions, the expansion velocity of the nebula in the recent past is approximately constant. While the proof of this is left as an exercise for the reader (to be done at the reader's own leisure and certainly not within this DRQ's time limit), an important implication is that the current radius of the nebula R obeys the relationship

$$R \approx X + vt,$$

where X denotes the extrapolated size of the nebula initially.

(x) [1 point] What is the value of X in meters?

Solution. Substituting the present-day values of R, v, and t, we obtain $X = 6.4\times10^{15}$ m.

(xi) [1 point] Assume that the Crab Nebula is stationary relative to us. Hence or otherwise, what is the angular size of the nebula now compared to 1758? Express your answer as a percentage.

Solution. Let A be the apparent size of the nebula. By inspection, it should be apparent that $A \propto R^2$. With

$$R_{704} = 6.4\times10^{15}\,\mathrm{m} + 1.6\times10^{6}(704\,\text{years}) = 4.2\,\mathrm{ly},$$

we have

$$\frac{A_{964}}{A_{704}} = \frac{R_{964}^2}{R_{704}^2} \approx \left(\frac{5.5}{4.2}\right)^2 = 171\%.$$

Recall that the apparent surface brightness of an object S has units of magnitude per square arcminute and is given by

$$S = m + 2.5\lg A,$$

where lg here indicates the base-10 logarithm $\log_{10}$, and A is the apparent size of the nebula. Assume that as in part (vii) that the nebula was visually Z times brighter in 1758 than in 2018.

(xii) [2 points] Calculate the change in average surface brightness since 1758, in terms of magnitude per unit area.

Solution. We have

$$S_{704} = m_{704} + 2.5 \lg A_{704},$$
$$S_{964} = m_{964} + 2.5 \lg A_{964},$$
$$\Delta S = \Delta m + 2.5 \lg \frac{A_{964}}{A_{704}}.$$

From the formula in the Formula Booklet,

$$\Delta m = 2.5 \lg \frac{L_{704}}{L_{964}} = -2.5 \lg \frac{L_{964}}{L_{704}} = 0.6825.$$

Therefore,

$$\Delta S = 0.6825 + 2.5 \lg \left(\frac{5.5}{4.2} \right)^2 = 1.268.$$

Mathematically speaking, if we compared an apparent unit area of the Crab Nebula between 1758 and 2018, the Crab Nebula now would have a surface brightness of 31% of what it had in in 1758. This is a noticeable change: it's like comparing the brightness of Vega to that of Deneb!

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Question 2
Are Black Holes Really Black?
(20 points)

Part 1: Gravitational Lensing **(Sub-total: 10 points)**

Gravitational lensing is a phenomenon where light from a distant source may be deflected by the curvature of space-time caused by a massive lensing object close to or in the line of sight between an observer and a distant object. We will take the lensing object to be a black hole in this problem. Take a look at the following illustration:

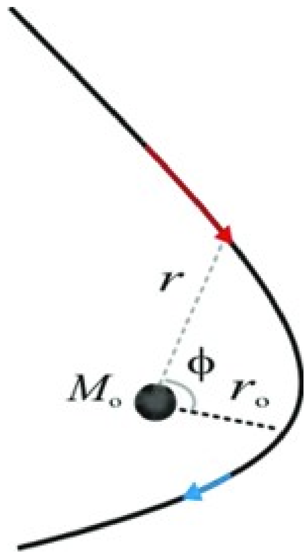


Figure 1: Path taken by light under the Schwarzschild metric. Note that the curvature in this diagram is greatly exaggerated.

The illustration above depicts the path that is taken by light (photon) under Schwarzschild metric, which is a metric that corresponds to the solution of Einstein's field equations with the assumption of chargeless and non-spinning perfectly spherical object and zero cosmological constant. It can be shown that the path can be described by the following equation in polar coordinate:

$$\frac{1}{r} = \frac{\cos\phi}{r_0} - \frac{r_s}{2r_0^2}\cos^2\phi + \frac{r_s}{r_0^2},$$

where r_s is the Schwarzschild radius of the black hole (i.e. the radius of the event horizon of the black hole), r_0 is the minimum distance from the path to the gravitating object, r is the distance from the object to a point along the path and ϕ is the angle between the line connecting the object with a point along the path and the line connecting the object with the point of closest approach.

(i) [2 points] By using Newtonian mechanics and equating the escape velocity at the Schwarzschild radius to the speed of light, derive the expression for the Schwarzschild radius, r_s, as given in the formula book. This derivation is heuristic: the steps are invalid, but it happens to be true. Can you explain why is it invalid?

Solution. We have

$$\frac{1}{2}mv_{\text{esc}}^2 - \frac{GMm}{r_s} = 0.$$

Since we have $v_{\text{esc}} = c$, hence we get

$$r_s = \frac{2GM}{c^2}.$$

It is invalid because Newtonian mechanics fails at the speed comparable to the speed of light, so our equation of the conservation of energy is not applicable to photons. The result just happens to be consistent with general relativity.

You may assume that $r_s \ll r_0$ such that the deflection of light from its original path is small.

(ii) [1 point] What is the value of ϕ (in radians) when the photon is very far and the mass of the black hole approaches 0?

Solution. With $r = \infty$ and $\frac{r_s}{r_0^2} = 0$, we have

$$\frac{\cos\phi}{r_0} = 0.$$

Hence, $\phi = \pm\frac{\pi}{2}$.

(iii) [1 point] Starting from the equation given, show that the path of the photon is a straight line if there is no black hole, as it should be.

Solution. We have

$$\frac{1}{r} = \frac{\cos\phi}{r_0},$$

i.e. $r_0 = r\cos\phi$. This is an equation of a straight line.

(iv) [2 points] Show that the total angular deflection of the photon can be expressed as

$$\delta = \frac{2r_s}{r_0}.$$

You may assume that when x is very small, we have $\sin x \approx \tan x \approx x$ and $\cos x \approx 1$.

Solution. Let δ be the deflection angle. Then half the deflection angle is $\frac{\delta}{2}$, then we have

$$\frac{\cos\left(\frac{\pi}{2} + \frac{\delta}{2}\right)}{r_0} + \frac{r_s}{r_0^2} = 0.$$

By a well-known trigonometric identity, we have

$$\cos\left(\frac{\pi}{2} + \frac{\delta}{2}\right) = -\sin\frac{\delta}{2}.$$

Hence,

$$-\frac{\delta}{2r_0} + \frac{r_s}{r_0^2} = 0,$$

i.e. $\delta = \frac{2r_s}{r_0}$.

(v) [2 points] Suppose we have an event where a black hole passes in front of a star such that the star, the black hole and Earth forms a straight line. This situation is modelled in the diagram below.

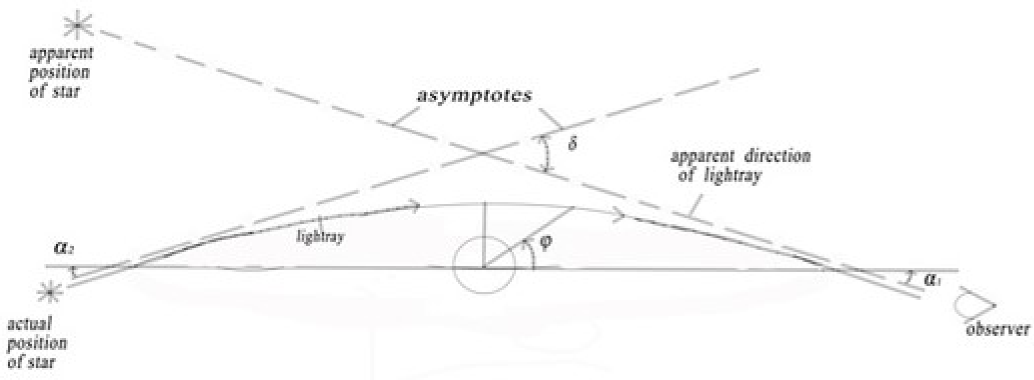


Figure 2: The scenario.

Suppose the black hole is located D_{BE} away from Earth and the distance from the black hole to the star is D_{SB}. Show that the following relation holds:

$$r_s = \frac{1}{2} r_0^2 \left(\frac{1}{D_{\text{BE}}} + \frac{1}{D_{\text{SB}}} \right).$$

(Hint: First, use the small angle approximation $\sin x \approx \tan x \approx x$. Then, note that if the angle is sufficiently small, the intersection point between two dashed lines should be really close to the point of closest approach.)

Solution. We have $\delta = \alpha_1 + \alpha_2$. Now, by small angle approximations,

$$\tan \alpha_1 \approx \alpha_1 \approx \frac{r_0}{D_{\text{BE}}},$$
$$\tan \alpha_2 \approx \alpha_2 \approx \frac{r_0}{D_{\text{SB}}}.$$

Therefore, we get

$$\delta = r_0 \left(\frac{1}{D_{\text{BE}}} + \frac{1}{\text{SB}} \right).$$

Therefore, we have

$$\frac{2r_s}{r_0} = r_0 \left(\frac{1}{D_{\text{BE}}} + \frac{1}{\text{SB}} \right),$$

so that

$$r_s = \frac{1}{2} r_0^2 \left(\frac{1}{D_{\text{BE}}} + \frac{1}{\text{SB}} \right).$$

(vi) [2 points] Suppose we have an event where a black hole located $7.5 \times 10^{20}\,\mathrm{m}$ away from Earth passes in front of a star $3.0 \times 10^{21}\,\mathrm{m}$ away from Earth. It is observed that the angular deflection of the star image observed from the Earth (often called the angular Einstein radius, denoted by α_1 in the figure) is $0.025\,\mathrm{arcseconds}$. Calculate the mass of the black hole in terms of solar masses.

Solution. First note that $r_s = \frac{2GM}{c^2}$ and $\frac{r_0}{D_{\mathrm{BE}}} = \alpha_1$. Hence by substituting into the equation of the previous part, we obtain

$$\frac{23GM}{c^2} = \frac{1}{2}\alpha_1^2 \frac{D_{\mathrm{BE}}}{D_{\mathrm{SB}}}(D_{\mathrm{BE}} + D_{\mathrm{SB}}).$$

Manipulating and substituting values, we obtain

$$\begin{aligned} M &= \left(\frac{\alpha_1 c}{2}\right)^2 \frac{D_{\mathrm{BE}}(D_{\mathrm{BE}} + D_{\mathrm{SB}})}{G D_{\mathrm{SB}}} \\ &= 4.95 \times 10^{33}\,\mathrm{kg} \\ &= 2490 M_\odot . \end{aligned}$$

Part 2: Hawking Radiation (Sub-total: 10 points)

If only classical physics is taken into account, black holes cannot emit radiation and their blackbody temperature can thus be considered to be zero. However, Stephen Hawking showed in his paper in 1974 that when quantum corrections are considered, black holes emit radiation according to the black-body spectrum. The corresponding black-body temperature is known as the Hawking temperature, T_H. For a Schwarzschild black hole of mass M, the Hawking temperature can be written as

$$T_H = \frac{\hbar c^3}{8\pi G k_B M},$$

where $\hbar$ is the reduced Planck's constant, G is the gravitational constant, and k_B is Boltzmann's constant.

According to the second law of thermodynamics, entropy of an isolated system never decreases. The definition of entropy S associated with a physical body is

$$dU = T\,dS,$$

where U is the internal energy of the body and T is the temperature of the body. In the case of a black hole, we may assume that the internal energy is the energy related to its mass (i.e. via Einstein's famous formula), and T is the black-body temperature.

(vii) [2 points] Show that the entropy of a black hole may be expressed as

$$S = KM^2,$$

where K is some constant. Write the value of this constant in terms of $c, G, \hbar$, and k_B.

Solution. We have $dS = \frac{dU}{T}$ and $dU = c^2\,dM$. Hence,

$$dS = s\frac{c^2\,dM}{\frac{\hbar c^3}{8\pi G k_B M}} = \frac{8\pi G k_B}{\hbar c} M\,dM.$$

When $M = 0$, there is no black hole and no entropy, hence $S = 0$. It follows that

$$\int_0^S ds = \frac{8\pi G k_B}{\hbar c}\int_0^M m\,dm,$$

i.e.

$$S = \frac{4\pi G k_B}{\hbar c} M^2.$$

But this is precisely the form needed, with $K = \frac{4\pi G k_B}{\hbar c}$.

A merger of two black holes was recently detected by Kip Thorne, Rainer Weiss, and Barry Barish, the winners of the 2017 Nobel Prize in Physics. The first detection of gravitational waves came from the merger of two black holes each with a mass equal to 30 solar masses, located at a distance of 1.3×10^9 ly from Earth.

(viii) [2 points] Assuming that the kinetic energy of the black holes are negligible compared to their rest energy when they collide, show that if two black holes with the same mass collide to make a bigger black hole, at most 30% of their initial rest energy can be converted to gravitational wave radiation.

(Hint: The entropy of an isolated system never decreases.)

Solution. Let S_i and S_f denote initial and final entropies respectively. We have

$$KM_f^2 = S_f \geq S_i = 2KM^2,$$

from which we obtain $M_f \geq \sqrt{2}M$. Hence, the maximum ratio of energy transferred to the gravitational waves is

$$r = \frac{2Mc^2 - \sqrt{2}Mc^2}{2Mc^2} = \frac{2-\sqrt{2}}{2} = 0.292\cdots < 0.30.$$

(ix) [1 point] Hence or otherwise, give an upper bound for the energy flux expected to pass through a detector situated on Earth in $\mathrm{mJ\,m^{-2}}$.

Solution. We have

$$\phi = \frac{E}{4\pi d^2} = \frac{2-\sqrt{2}}{2}\frac{2Mc^2}{2\pi d^2} \approx 1.65\,\mathrm{mJ\,m^{-2}}\,.$$

We will now consider black hole evaporation. Assuming that there is no in-falling matter or energy, a black hole will slowly radiate away its mass through Hawking radiation. Although a correct treatment of the evaporation process at high energy scales requires a theory of quantum gravity, as long as T_H is below the Planck scale, the semi-classical approach we have been using so far suffices. In what follows, we will obtain an estimate of the black hole evaporation timescale; this will be an estimate for the lower bound on the evaporation process duration.

You are given that the value of Stefan-Boltzmann constant is

$$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}.$$

(x) [1 point] Determine the power radiated by a black hole as a function of mass, assuming that the effective area is the area of the event horizon.

Solution. We have

$$\begin{aligned} P &= \sigma T_H^4 (4\pi r_s^2) \\ &= \frac{\pi^2 \kappa_B^4}{60\hbar^3 c^2} \left(\frac{\hbar c^3}{8\pi G k_B M} \right)^4 \left(4\pi \frac{4G^2 M^2}{c^4} \right) \\ &= \frac{\hbar c^6}{15360\pi G^2 M^2}. \end{aligned}$$

(xi) [2 points] Hence, or otherwise, calculate the evaporation time of a black hole with the same mass as the Sun in seconds. Comment on the result with a suitable comparison.

Solution. We have $P = -\frac{dU}{dt}$. Hence,

$$-\frac{\hbar c^6}{15360\pi G^2 M^2} = \frac{dU}{dt} = \frac{c^2\, dM}{dt}.$$

Let the time for complete evaporation be t'. Then,

$$t' = \int_0^{t'} dt = -\frac{15360\pi G^2}{\hbar c^4} \int_M^0 m^2\, dm = \frac{5120\pi G^2}{\hbar c^4} M^3 = 6.6 \times 10^{74}\,\mathrm{s}\,.$$

This is much larger than the age of the universe.

Next, consider a black hole exposed to the Cosmic Microwave Background (CMB) radiation. You may assume that the CMB radiation is a black-body radiation with a temperature $T_{\mathrm{CMB}} \approx 2.8\,\mathrm{K}$ that fills the entire universe. The black hole will therefore lose energy from Hawking radiation and absorb energy from the CMB radiation according to the Stefan-Boltzmann law.

(xii) [1 point] When a black hole reaches a certain critical mass M_C, the net power radiated by the black hole is exactly zero. This is often called the equilibrium condition. Calculate this critical mass in kg. Name an astronomical object which has a similar mass, up to one order of magnitude.

Solution. Clearly, the equilibrium occurs when $T_H = T_{\mathrm{CMB}}$, i.e.

$$T_{\mathrm{CMB}} = \frac{\hbar c^3}{8\pi G k_B M_C}.$$

Then,

$$M_C = \frac{\hbar c^3}{8\pi G k_B T_{\text{CMB}}} = 4.38 \times 10^{22}\,\text{kg}\,.$$

This is approximately 0.6 times the mass of the moon.

(xiii) (a) [0.5 points] Is the equilibrium mentioned above in Part (xii) stable or unstable?

(b) [0.5 points] Comment on the value obtained in Part (xi) in relation to the above conclusion.

Solution. Note that when $M > M_C$, $T_H < T_{\text{CMB}}$, so all black holes with mass larger than M_C will have temperature lower than the CMB radiation. This means that it will gain energy from the CMB radiation, resulting in an increasing mass. Conversely, when $M < M_C$, $T_H > T_{\text{CMB}}$, and thus the black hole will evaporate. We conclude that the equilibrium is unstable.

This means that the time for a black hole to evaporate will be even greater than what we calculated in Part (xi), since we need to wait for the CMB to cool down until $T_H > T_{\text{CMB}}$ for the black hole to start evaporating.

Question 3
The Red Distance
(Total: 20 points)

Introduction

It is a well-known fact that nothing travels faster than light (except for fictional objects like the Starship Enterprise). Consequently, in light of the age of the universe being approximately 13.7 billion years, it comes as a surprise to most first-time astronomers to learn that the radius of the observable universe is approximately 46.5 Gly.

Of course, astronomers like those in this competition know that this discrepancy is due to the expansion of space. In this question, we will look at how expansion of space affects our intuitive notion of distance.

Part 1: Relativistic Redshift (Sub-total: 7 points)

Distances of objects near the edge of the observable universe are often difficult to measure precisely, due to observational limitations. For this reason, amongst others, distances from Earth to such objects are often measured in terms of their *relativistic redshifts*. The relativistic redshift z of a moving object in the radial direction is given as

$$1 + z = \gamma \left(1 + \frac{v}{c}\right),$$

where γ is the Lorentz factor and v is the velocity of the object moving away from the observer in the radial direction.

(i) [2 points] There are three main types of redshifts, but the redshift we are primarily concerned with in this question is *cosmological redshift*. Explain the phenomena of cosmological redshift.

Solution. Cosmological redshift is the phenomenon of whereby distant cosmological objects are perceived to have a redshift. This redshift is due to the perceived receding velocity due to the expansion of the universe stretching the wavelength of light.

Using this formula, one can determine "distances" from the observer to the object. This is done by interpreting cosmological redshift as a receding velocity. That is to say, suppose we had an object at an extreme distance. To this distance we may associate a "rate at which the object is moving away from us" due to the expansion of space. This rate of recession is expressed as a receding velocity, and we may utilise the relativistic redshift formula to compute the proper distance from the object to us.

(ii) [3 points] Assume that a distant object is completely stationary, such that any perceived motion is solely due to cosmological expansion. Prove that the (proper) distance d from the observer to the object is given by

$$d = C \cdot \frac{(1+z)^2 - 1}{(1+z)^2 + 1},$$

where C is some expression in terms of known quantities to be determined. In your answer, you should state an expression for C, with reference to the Formula Booklet, or otherwise.

Solution. Since the object is completely stationary, it suffices to consider the receding distance due to universal expansion. Recall that $v = H_0 d$, where H_0 is the Hubble parameter today. Let $\frac{H_0}{c} = k$. Then, expanding the given equation, we have

$$1 + z = \frac{1}{\sqrt{1 - k^2 d^2}}(1 + kd) = \sqrt{\frac{1 + kd}{1 - kd}}.$$

Squaring,

$$(1 + z)^2 = \frac{1 + kd}{1 - kd}.$$

Then we can rearrange and obtain

$$(1 + z)^2 - (1 + z)^2 kd = 1 + kd,$$

from which we obtain

$$d = \frac{1}{k} \frac{(1 + z)^2 - 1}{(1 + z)^2 + 1},$$

so that $C = \frac{1}{k} = \frac{c}{H_0}$. The proof is complete.

(iii) [2 points] Hence or otherwise, explain why C cannot remain constant with time. Compute the value of C today based on the formula booklet.

Solution. Suppose C remained constant with time. This means H_0 would be constant with time. Then the equation $v = H_0 d$ would be constant with time, i.e. for fixed d the rate of expansion is constant with time. Clearly this is in contradiction with the accelerating rate of expansion of the universe.

Alternatively, since $H = \frac{a'(t)}{a(t)}$, H cannot remain constant with time as the expansion of the universe is accelerating.

In either explanation, the key point is an *accelerating* expansion.

The value is

$$C = \frac{c}{H_0} \approx 4.42\,\text{Gpc}\,.$$

(Note: Accept a range from 4.37 Gpc to 4.47 Gpc, due to uncertainty in provided H_0 value.)

Part 2: Let's Hubble Along (Sub-total: 4 points)

The concept of measuring a "distance" takes a rather interesting turn, pun intended, when we approach the edge of the observable universe. To further define distances in a proper fashion, however, we first need to consider the Hubble parameter H. The Hubble parameter is a function of the redshift z, and so we write it $H(z)$ without loss of generality. The derivation of the Hubble parameter comes from Einstein's field equations, but we save you all this work and give you the final result. It is given by

$$(H(z))^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3},$$

where G is the gravitational constant, ρ is the mass density of the universe, k is the normalised spatial curvature of the universe, Λ is the cosmological constant, and $a = \frac{1}{1+z}$ is the scale factor. This is also known as Friedmann's second equation.

A bit needs to be said about the scale factor a. It relates the proper distance (see Part 3 for an explanation) between two objects. If at the present time we receive light from a distant object with redshift z, the scale factor at the time the light originated from the object is $a = \frac{1}{1+z}$.

(iv) [1 point] Explain why the relation $d(t) = a(t)d_0$ holds, where $d_0 = a(t_0)$ is the distance at some reference time, and t is time taken with reference to t_0.

Solution. This is simply a mathematical fact. At a time $t > t_0$, the distance between two objects has changed (i.e is scaled) by precisely the scale factor $a(t)$. So $d(t) = a(t)d_0$.

(v) [3 points] Suppose that the universe is matter-dominated with density of matter today at ρ_0. It is given that

$$\rho_c = \frac{3H_0^2}{8\pi G}, \qquad \Omega_m = \frac{\rho_0}{\rho_c}, \qquad \Omega_k = -\frac{kc^2}{H_0^2}, \qquad \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2}.$$

Here, H_0 is the Hubble constant today. Show carefully that

$$H(z) = H_0 E(z),$$

where

$$E(z) = \sqrt{B(1+z)^3 + C(1+z)^2 + D},$$

and B, C, D are constants to be determined in terms of the four quantities provided above.

Solution. Since we assume the universe is matter-dominated, the mass density of the universe can be simply taken to be the density of matter multiplied by the inverse of the cube of the scale factor, so that

$$\rho = \frac{\rho_0}{a^3}.$$

Then, we have

$$\rho = \frac{\rho_c \Omega_m}{a^3} = \frac{3H_0^2 \Omega_m}{8\pi G a^3}.$$

Now, by direct manipulation we have

$$-kc^2 = H_0^2 \Omega_k$$

and

$$\Lambda c^2 = 3H_0^2\Omega_\Lambda.$$

By direct substitution into Friedmann's second equation, then we have

$$\begin{aligned}(H(z))^2 &= \frac{H_0^2\Omega_m}{a^3} + \frac{H_0^2\Omega_k}{a^2} + H_0^2\Omega_\Lambda \\ &= H_0^2(\Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda).\end{aligned}$$

Taking square roots yields

$$H(z) = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda},$$

so $B = \Omega_m, C = \Omega_k, D = \Omega_\Lambda$, and $E(z) = \sqrt{\Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda}$.

Part 3: Too Far Away! **(Sub-total: 9 points)**

There are three types of distances we shall now contend with: the *proper distance*, the *comoving distance*, and the *light travel distance*.

The proper distance between two objects is the separation of the two objects measured at a specific cosmological time. Loosely speaking, it is the distance between the objects factoring in cosmological expansion. The proper distance changes with time.

The comoving distance, on the other hand, is the separation of the two objects measured at the current cosmological time. Loosely speaking, it is the distance between the objects factoring out cosmological expansion. The comoving distance does not change with time. To define a comoving distance, we need to fix a time T and measure proper distance at that time T. That becomes the comoving distance.

In short, while proper distance can be likened to a movie, the comoving distance can be likened to taking a frame, or a snapshot, of an instant in the said movie.

The light travel distance is the time taken for light to reach from the object to us, multiplied by the speed of light.

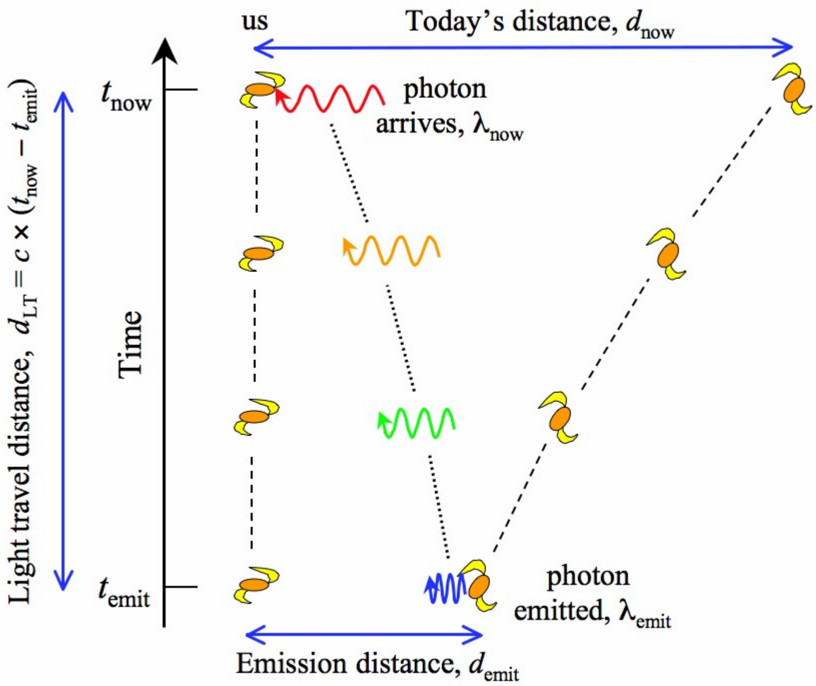


Figure 3: A nice illustration of the three distances we found on Google. Here, emission distance is the comoving distance with respect to t_{emit}, and also the proper distance at that time. Today's distance is the proper distance today.

(vi) [1 point] Using a simple example, or otherwise, demonstrate that light travel distance is distinct from the proper distance.

Solution. For instance, consider an object perhaps 13.3 billion years old. Light took 13.3 billion years to arrive, so the light travel distance is 13.3 Gly. But the proper distance is very much further (relatively near the edge of the observable universe).

(vii) [4 points] Suppose the Hubble constant today is H_0. Let $d_H = \frac{c}{H_0}$ be the Hubble distance. Show that the comoving distance d_C satisfies

$$d_C(Z) = d_H \int_0^Z \frac{1}{E(z)}\, dz,$$

where $d_C(Z)$ is the comoving distance of an object with cosmological redshift Z.

(Hints: One way to do this is to start by considering expressing d_C using c and a. Note that for a given object, the redshift Z is a function of time, and vice-versa (since space is expanding). At some point in your answer, you may find the relation $H(t) = \frac{a'(t)}{a(t)}$ useful.)

Solution. The RHS is equivalent to

$$\int_0^Z \frac{c}{H(z)}\, dz.$$

We consider photons. Let a be the scale factor. The comoving distance can then be expressed as an integral over time for which photons travel to us, i.e.

$$d_C(Z) = \int_0^t \frac{c}{a(t)}\, dt.$$

Consider, then, that $1 + z = \frac{1}{a(t)}$. With the hint, we have

$$dz = -\frac{1}{a(t)^2} a'(t) = -H(t)\frac{1}{a(t)}\, dt.$$

By substitution,

$$d_C(Z) = \int_0^Z \frac{c}{H(t)}\, dz.$$

Rewriting H in terms of z completes the proof, since $H(z) = H_0 E(z)$.

(viii) [3 points] Find a similar expression for the light travel distance $d_T(Z)$.

(Hint: At some point in your answer, you may find the following rule useful. If $f(x) = \int_0^x g(s)\, ds$, then $f'(x) = g(x)$.)

Solution. The answer is

$$d_T(Z) = d_H \int_0^Z \frac{1}{(1+z)E(z)}\, dz.$$

The idea is similar to the above. We consider the radial distance. The light travel distance is given by

$$d_T(Z) = \int_0^r \frac{a(t)}{c}\, dr.$$

(This is done by considering photons again travelling along the radial direction. One obtains the relation

$$\frac{c}{a(t)}\, dt = -\, dr,$$

where the negative sign simply implies directionality towards the origin.)

Notice that r in this instance is the comoving distance. That is, $r(z) = d_C(z)$; it's just different ways of writing the same quantity. Now, by rewriting the integrating variable from the previous part, we have

$$r(z) = \int_0^z \frac{c}{H(\zeta)}\, d\zeta,$$

from which we obtain by the hint that

$$dr = \frac{c}{H(z)}\, dz.$$

With this substitution,

$$d_T(Z) = \int_0^r \frac{a(t)}{c}\, dr = \int_0^Z \frac{a(t)}{c} \frac{c}{H(z)}\, dz = \int_0^Z \frac{1}{(1+z)E(z)}\, dz.$$

(ix) [1 point] Explain why the age of the universe is given as

$$\lim_{z\to\infty} \frac{d_T(z)}{c}.$$

Solution. As we approach the edge of the observable universe, $z \to \infty$. The light travel distance to the edge is $\lim_{z\to\infty} d_T(z)$, and so the time taken is $\lim_{z\to\infty} \frac{d_T(z)}{c}$, i.e. this is the age of the universe.

Question 4
A Study of the Big Dipper
(Total: 20 points)

Part 1: Introduction **(Sub-total: 8 points)**

The Big Dipper is well known across many cultures since antiquity. Some details about the Big Dipper are provided below.

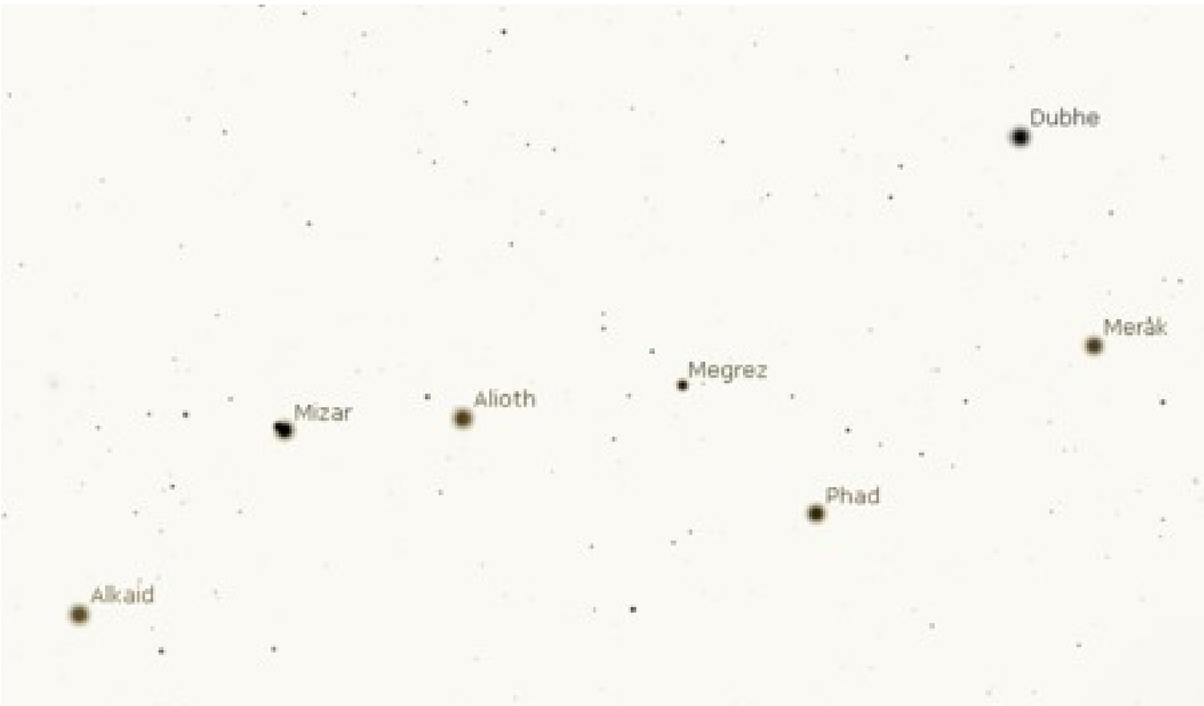


Figure 4: The Big Dipper.

Star	Apparent Magnitude m	Right Ascension	Declination
Dubhe	+1.79	11h 03min	$+61^\circ\ 45'$
Merak	+2.37	11h 01min	$+56^\circ\ 22'$
Phad	+2.44	11h 53min	$+53^\circ\ 41'$
Megrez	+3.31	12h 15min	$+57^\circ\ 01'$
Alioth	+1.77	12h 54min	$+55^\circ\ 57'$
Mizar	+2.27	13h 23min	$+54^\circ\ 55'$
Alkaid	+1.86	13h 47min	$+49^\circ\ 18'$

Table 2: Stars of the Big Dipper.

(i) Modern astronomers label the Big Dipper as an *asterism*.

(a) [1 point] Define an asterism.

(b) [1 point] State an example of an asterism other than the Big Dipper.

Solution. An asterism is simply a pattern of stars in the night sky.

There are a number of other asterisms, e.g. Winter Hexagon, Northern Cross, Summer Triangle, and so on.

(Note: Accept any reasonable answer.)

(ii) [2 points] It is well known that one can find Polaris simply by extending a line from Merak to Dubhe. This is merely one of the ways where the Big Dipper serves as an important signpost to other bright stars. Explain how one can use the Big Dipper to find two other bright stars (except Polaris).

Solution. The handle of the Big Dipper arcs to Arcturus and then speeds on to Spica.

Alternatively, a line from Megrez to Phad leads to Regulus (but this is not so easy to remember).

(Note: Accept any reasonable answers.)

(iii) The Big Dipper is placed in the far northern sky, making it difficult for southern observers to see. Below is a list of four cities in Australia and New Zealand.

City	**Latitude**	**Longitude**
Hobart, AU	$42^\circ\,52'\,\mathrm{S}$	$149^\circ\,19'\,\mathrm{E}$
Brisbane, AU	$27^\circ\,28'\,\mathrm{S}$	$153^\circ\,02'\,\mathrm{E}$
Auckland, NZ	$36^\circ\,50'\,\mathrm{S}$	$174^\circ\,44'\,\mathrm{E}$
Wellington, NZ	$41^\circ\,17'\,\mathrm{S}$	$174^\circ\,46'\,\mathrm{E}$

Table 3: Cities in Australia and New Zealand.

(a) [1.5 points] Which of these cities cannot see any star of the Big Dipper?

(b) [1.5 points] Which of these cities can only see part of the Big Dipper over the course of an entire day?

In answering the above two parts (a) and (b), you should show your working. You may ignore atmospheric extinction and assume a flat horizon.

Solution. The southernmost star of the Big Dipper is Alkaid. With a declination of $+49^\circ\,18'$, it never rises south of $(90^\circ\,0' - 49^\circ\,18')\,\mathrm{S} = 40^\circ\,42'\,\mathrm{S}$. This means that the Big Dipper is invisible from Hobart, AU and Wellington, NZ.

It is easier to first consider which cities can see all of the Big Dipper. Dubhe, the northernmost star of the Big Dipper, never rises south of $(90^\circ\,0' - 61^\circ\,45')\,\mathrm{S} = 28^\circ\,15'$. This means that Brisbane, AU can see all of the Big Dipper. Thus, out of the 4 cities, only Auckland, NZ is able to see a partial Big Dipper.

(iv) [1 point] To the naked eye, the stars of the Big Dipper have noticeably different brightnesses. How much brighter is the brightest star of the Big Dipper compared to its faintest star? Express your answer in percentages.

Solution. From the table, the brightest (Alioth) and faintest (Megrez) stars have magnitudes of +1.77 and +3.31 respectively. The formula sheet contains a formula linking luminosity and absolute magnitude; that same relationship links brightness and apparent magnitude. Therefore,

$$\frac{B_1}{B_2} = 10^{\frac{m_1 - m_2}{2.5}} = 10^{\frac{3.31-1.77}{2.5}} = 4.13 = 413\%.$$

Part 2: Starry Night Over the Rhône **(Sub-total: 12 points)**

The Big Dipper is prominently featured in Vincent Van Gogh's *Starry Night Over the Rhône*. Painted in September 1888, it features the city of Arles, France at night.

Figure 5: Van Gogh's *Starry Night Over the Rhône*. Truly a work of art.

But what time is it? Using the night sky depicted here, we can recover the time of night that Van Gogh was trying to faithfully depict.

(v) [2 points] By matching the features of the cityscape with actual landmarks, we know that this was painted from Arles at the coordinates $43° \, 41' \, \mathrm{N}$, $4° \, 38' \, \mathrm{E}$. From this point, the Big Dipper is circumpolar. What is the lowest altitude reached by any part of the Big Dipper when it skims the horizon?

Solution. It should be obvious that the southernmost star (Alkaid) will have the lowest possible altitude. There is a formula to obtain this altitude, but it is far more instructive to explain the underlying process in words.

- We know the altitude of the North Celestial Pole. It is located at $43° \, 41'$.
- Alkaid is $90° \, 0' - 49° \, 18' = 40° \, 42'$ south of the North Celestial Pole
- Thus, the lowest altitude occurs when Alkaid lies on the lower local meridian, directly below the North Celestial Pole. At this point, its altitude is $43° \, 41' - 40° \, 42' = 2° \, 59'$.

(vi) For simplicity, let us assume that this work was painted on the autumnal equinox (September 22).

(a) [1 point] On this date, what is the approximate right ascension of the Sun to the nearest minute?

(b) [1 point] Briefly explain your answer.

Solution. Recall that 0h 00min is defined by the position of the Sun at the vernal equinox (i.e. the first point of Aries). Since exactly half a year separates the autumnal equinox from the vernal equinox, the right ascension of the Sun then is simply $180°$ away from 0h 00min, i.e. it lies at 12h 00min.

(vii) [2 points] Define the hour angle as the number of hours since the object passed the upper local meridian. This means that when an object is at the highest point in the sky, it has an hour angle of 0h 00min.

With this in mind, estimate the hour angle of Merak/Dubhe to the nearest minute. In your answer, you should state a suitable assumption/simplification that you need to make.

Solution. Since the line connecting Merak/Dubhe is almost exactly perpendicular to the horizon, let us assume Merak/Dubhe lies on the lower local meridian (i.e. their lowest possible altitude). This implies that their hour angle is 12h 00min.

For the rest of this DRQ, you may treat Merak and Dubhe as sharing the same right ascension of 11h 02min.

(viii) [1 point] Hence or otherwise, determine the hour angle of the Sun during the moment depicted in this painting.

Solution. If the Sun has a known RA of 12h 00min, we know that it rises 58 minutes later than Merak/Dubhe, since Merak and Dubhe have a known RA of 11h 02min. Thus, if we know the hour angle (HA) of Merak/Dubhe is 12h 00min, the Sun must be 58 minutes behind, giving us a HA of 11h 02min.

(ix) [2 points] For simplicity, let us define sunset as the point in time when the apparent centre of the Sun touches the horizon, ignoring atmospheric refraction. In Greenwich, UK, sunset on September 22nd occurs at 1753h (Greenwich Mean Time, a.k.a. GMT). When would sunset be if we were displaced $4^\circ\,38'$ E from Greenwich? Give your answer in GMT.

Solution. If the Earth rotates 360° degrees in a day, it takes

$$\frac{4^\circ\,38'}{360\circ} \times 24\,\text{hours} = 18.5\,\text{minutes}$$

to rotate by $4^\circ\,38'$.

Since the new location lies east of Greenwich, sunset occurs earlier than Greenwich. Thus, the sunset occurs 18.5 minutes before 1753h, giving us 1734h or 1735h, depending on how you round your answer.

(x) [2 points] An astute observer notes that Arles and Greenwich are at different latitudes. Hence, the answer found in Part (ix) may not necessarily correspond to the actual sunset time at Arles in GMT. Is this a major concern? Explain your answer.

Solution. No, it is not.

Recall that it is currently the autumnal equinox. Since the Sun is on the celestial equator, the length of day is approximately the same around the world (excluding pathological cases like the poles).

Due to these special circumstances, sunset in Greenwich does not occur appreciably earlier/later than in Arles, after adjusting for the difference in longitude.

(xi) [1 point] Assume your answer in Part (ix) is correct. Hence, find the exact time depicted in this painting, in GMT.

Solution. Since the Sun is on the celestial equator, when the Sun set, it had a HA of 6h 00min. At the time depicted in the painting, the sun had a HA of 11h 02min. This means the scene depicted here occurs 5h 02min after sunset, or around 2236h or 2237h.

Because the proof is in the pudding, here is the result when you key in the exact date, time and location in Stellarium.


Earth, Arles, 24m
FOV 52.4°
72.6 FPS
1888-09-22
22:36:00 UTC+00:00

Question 5
Alice and Bob Circling in the Sky
∼A Story of Binary Star Systems∼
(Total: 20 points)

Introduction

In a faraway land in outer space, in the constellation of Gemini lies the two stars, Alice and Bob. This is a love story of Alice and Bob, two binary stars circling in the heavens above. From our astronomical observations, it is found that most stars indeed do come in pairs (as most good things do), unlike our lonely parent star, the Sun. The study of these binary systems is hence an important research in understanding the formation of star systems in nebulae.

In the following questions, you may assume that the plane of orbit is parallel to the plane of observation.

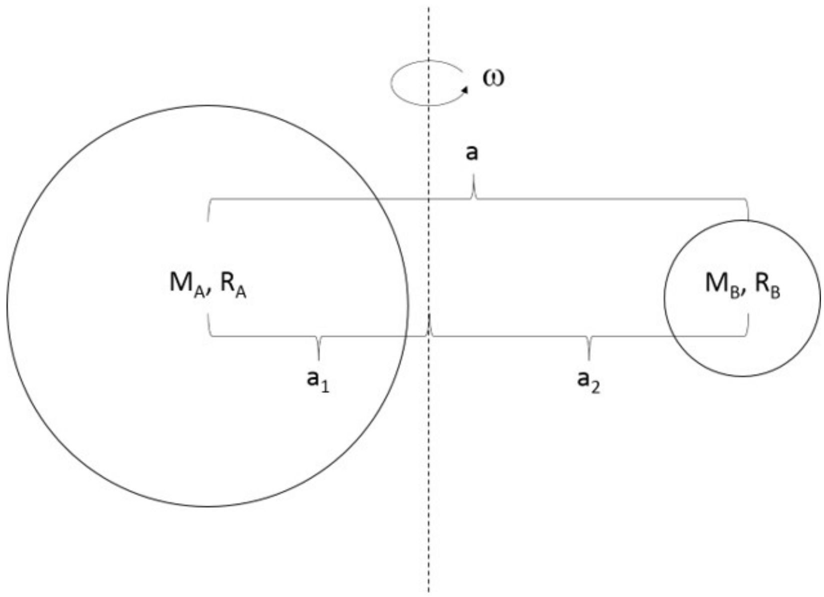


Figure 6: Alice and Bob (not drawn to scale) circling.

Part 1: Celestial Mechanics (Sub-total: 9 points)

Let Alice and Bob have mass M_A and M_B, and radius R_A and R_B respectively, as shown in the diagram (the subscripts A and B refers to Alice and Bob respectively). Note that Alice and Bob have different masses $M_A > M_B$.

(i) [2 points] The distance to the centre of mass as measured from the primary star Alice is a_1. Express a_1 in terms of the distance a between the two stellar centres, M_A, and M_B.

Solution. We have the system

$$\begin{aligned} a_1 + a_2 &= a, \\ \frac{GM_AM_B}{a^2} - M_Aa_1\omega^2 &= 0, \\ \frac{GM_AM_B}{a^2} - M_Ba_2\omega^2 &= 0. \end{aligned}$$

From this it follows that

$$M_Aa_1 = M_Ba_2 = M_B(a - a_1),$$

so we get

$$a_1 = \frac{M_Ba}{M_A + M_B}.$$

Consider the following graph of radial velocity against phase of Alice and Bob.

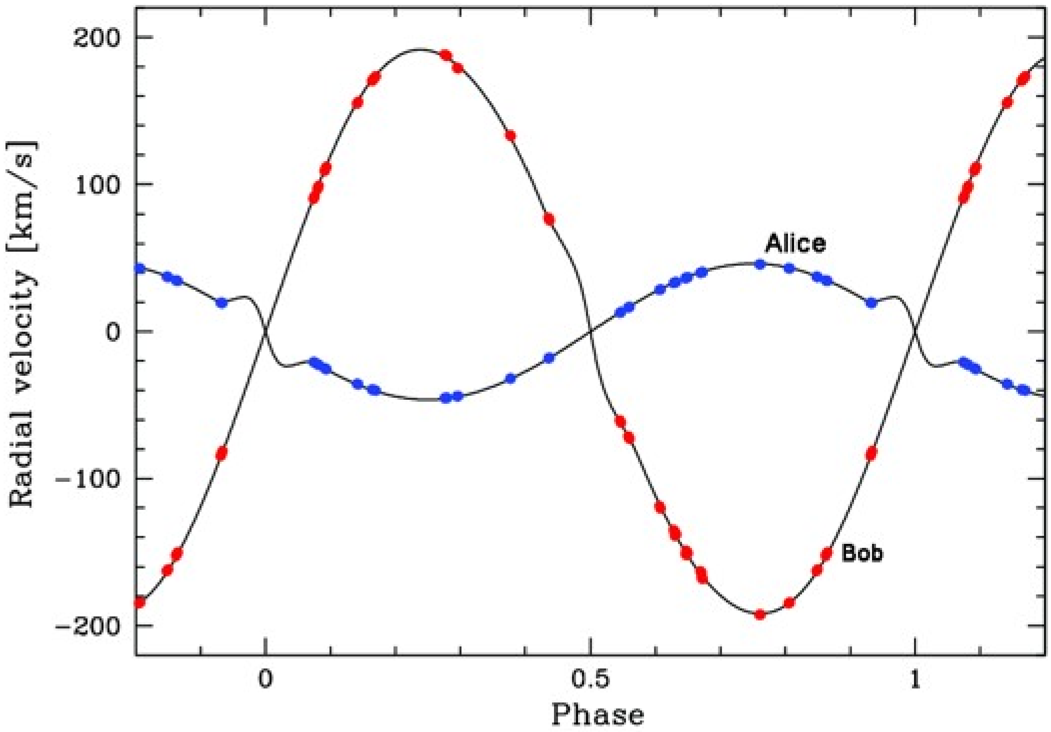


Figure 7: Alice and Bob's phase graph.

It is found from observation that the distance between the two stars (centre of mass to centre of mass) is 0.062 AU.

(ii) [3 points] Determine the masses M_A and M_B of Alice and Bob respectively in solar units.

Solution. We have

$$\frac{v_A}{v_B} = \frac{M_B}{M_A} = \frac{a_1}{a_2} = \frac{40 \times 10^3}{200 \times 10^3} = \frac{1}{5}.$$

Furthermore, $r = 0.062 \times 1.5 \times 10^{11}$. We also have that $a_1 + 5a_2 = a$. Also note that

$$M_\odot = 2 \times 10^{30},$$
$$R_\odot = 6.69 \times 10^8.$$

Hence,

$$\omega = \frac{240 \times 10^3}{r}.$$

Now,

$$M_B = \frac{1}{G} \times \frac{r^3\omega^2}{6} \times \frac{1}{M_\odot}$$

Also, $M_A = 5M_B$. Hence, $M_B = 0.669M_\odot$ and $M_A = 3.346M_\odot$.

(iii) [3 points] It was found that the surface temperature of Alice and Bob are 13000 K and 4500 K respectively. Using the results in Part (ii) and the Stefan-Boltzmann law, determine the radius R_A and R_B of Alice and Bob in solar units. State all assumptions used in your calculations.

Solution. Assuming that Alice and Bob are main sequence stars,

$$L \propto M^{3.5}.$$

Hence,

$$\begin{aligned}
\frac{L_A}{L_\odot} &= \left(\frac{R_A}{R_\odot}\right)^2 \left(\frac{T_A}{T_\odot}\right)^4 = 3.346^{3.5}, \\
\frac{L_B}{L_\odot} &= \left(\frac{R_B}{R_\odot}\right)^2 \left(\frac{T_B}{T_\odot}\right)^4 = 0.669^{3.5}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{R_A}{R_\odot} &= \sqrt{3.346^{3.5} \left(\frac{T_\odot}{T_A}\right)^4} = 1.64, \\
\frac{R_B}{R_\odot} &= \sqrt{0.669^{3.5} \left(\frac{T_\odot}{T_B}\right)^4} = 0.816.
\end{aligned}$$

(iv) [1 point] Why can we use the Stefan-Boltzmann law in our calculation?

Solution. The Stefan-Boltzmann law expresses the relationship between the luminosity of a spherical black-body and its temperature, while a star is assumed an almost perfect black-body. Hence, we can use the Stefan-Boltzmann law.

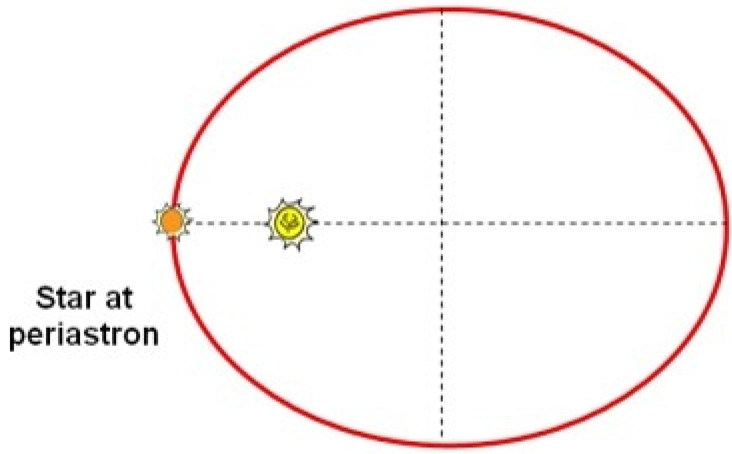
Part 2: Gravitational Waves **(Sub-total: 11 points)**

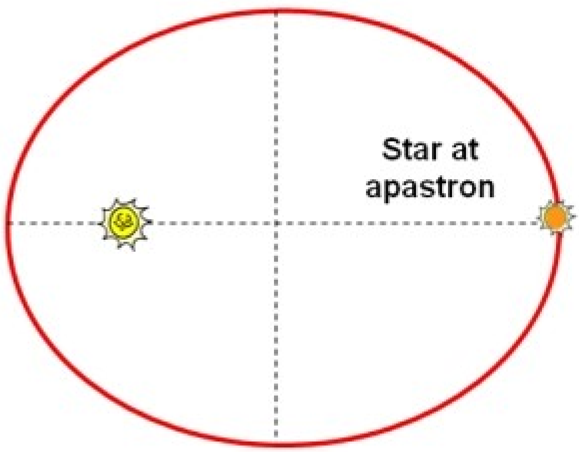
In another universe (if you buy into the idea of the multi-verse), Alice and Bob are in fact two neutron stars that are spiralling into each other. The story of Alice and Bob could develop into either of two fates: they merge to form a new neutron star, or they merge and become a black hole. Understanding the fate of the merger requires us to develop an understanding of the in-falling process.

Data of Alice and Bob was collected from ground base observatories, monitoring their dynamical relationship. It is understood that due to the densities of the neutron stars, the orbit has an initial period P_0 of approximately 7.752 hours. Furthermore, Alice is in fact a pulsar, emitting strong electromagnetic radiation in the radio-wave spectrum. We observe instances that Alice returns to the periastron, and call that the *Time of Periastron Passage*. Given that the orbit is decaying, the period P_n after n periastron passages is smaller than the initial value P_0.

(v) [2 points] The periastron refers to the point of closest approach between the binary stars. It is equivalent to the perigee of a satellite (natural or man-made) orbiting the planet. Using this information, sketch a diagram depicting the orbit of the two neutron stars. In your diagram, you should label the periastron and apastron.

Solution. Note: The below diagrams are to be combined into a single diagram in your answer. They are split here for clarity purposes.





(vi) [1 point] Suggest a possible cause of the orbital decay.

Solution. Given that they are neutron stars, the cause of orbital decay is naturally due to the emission of gravitational waves (Hulse and Taylor).

From theoretical calculations, the time between the 0^{th} and N^{th} periastron, T_N, can be expressed as an equation

$$\frac{T_N}{P_0} = \sum_{i=1}^{N} (1 + \dot{P})^i.$$

Note that $\dot{P}$ is the time derivative of P, the orbital period.

(vii) [2 points] Expand the summation and show that given that $\dot{P}$ is extremely small, the summation may be approximated as

$$\frac{T_N}{P_0} = N + \frac{N(N+1)}{2}\dot{P}.$$

Solution. We may ignore $\dot{P}$ terms of power 2 and higher, since $\dot{P}$ is very small. Hence,

$$\begin{aligned}\frac{T_N}{P_0} &= \sum_{i=1}^{N} (1 + \dot{P})^i \\ &\approx \sum_{i=1}^{N} (1 + i\dot{P}) \\ &= N + \dot{P}\sum_{i=1}^{N} i \\ &= N + \frac{N(N+1)}{2}\dot{P}.\end{aligned}$$

where in the last line we make use of the identity for $1 + 2 + \cdots + N$.

(viii) [6 points] Hence, the change in the orbital period after i periastrons is

$$\Delta t_i = T_i - T_0 = \frac{i(i+1)}{2} P_0 \dot{P}.$$

Complete Table 4, linearise the above equation, and plot an appropriate graph to obtain the observed value of $\dot{P}$.

(*Note: You should detach the table provided and attach it to your answer script.*)

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Detach this page and attach it to your answer script.

Approximate Date	**Time of Periastron Passage (JED – 2,440,000)**	**Number of Hours Between 2 Observations**	**Number of Complete Orbits Between 2 Observations**	**Actual Cumulative Completed Orbits**	Δt_i
1974.77	2331.446	0	0	0	0
1974.93	2389.586	1395.349	180	180	−0.000883681
1976.13	2826.924	10496.126	1354.000036		−0.063867225
1976.93	3118.591	7000.001	903.0000096		−0.161151198
1977.58	3356.640	5713.179	737.0000111		−0.27333482
1977.96	3493.591	3286.822	424.0000088		−0.351226531
1978.23	3593.397	2395.349	309.0000019		−0.414135323
1978.42	3663.488	1682.171	217.0000065		−0.46140991
1978.82	3807.544	3457.365	446.0000028		−0.566593685
1979.31	3988.423	4341.086	560.0000059		−0.713943244
1980.10	4276.537	6914.730	892.0000106		−0.983779849
1980.59	4454.509	4271.319	551.0000043		−1.172027064
1980.59	4455.477	23.256	3.000000635		−1.173097083
1981.14	4656.382	4821.706	622.000003		−1.405491826

Table 4: Table for Part (viii).

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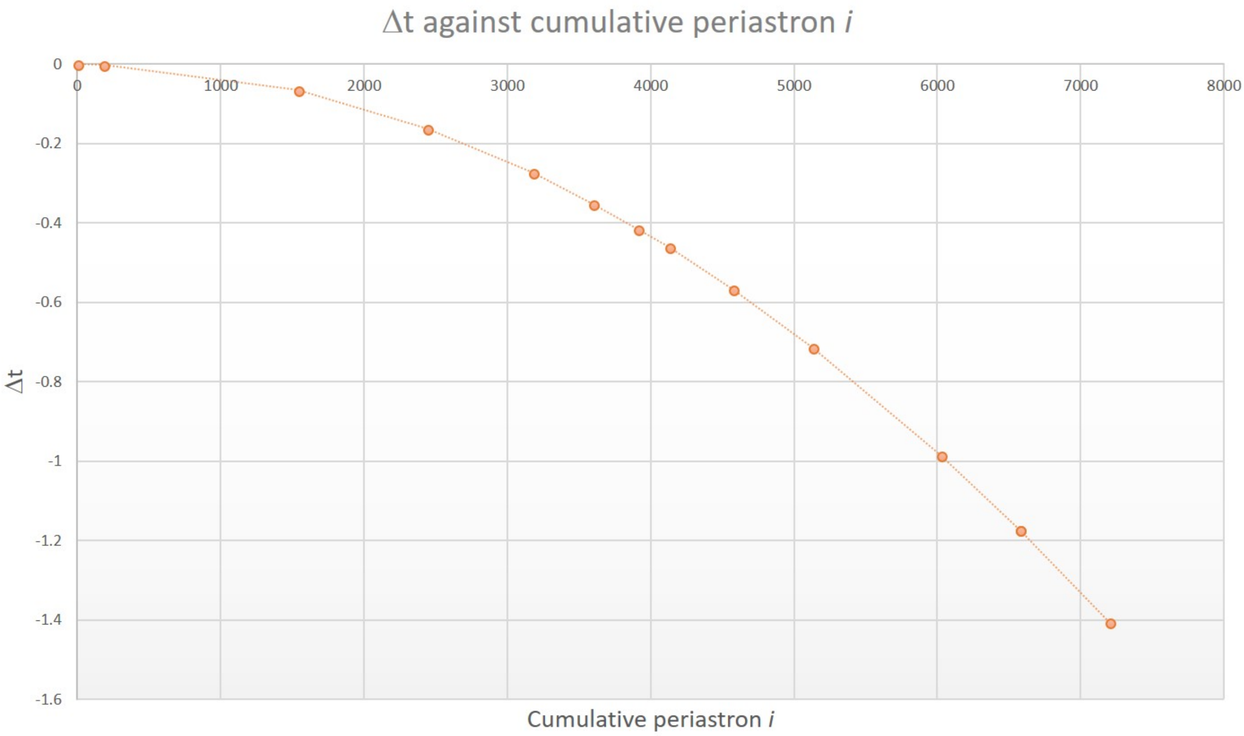
Solution. The linearised equation is

$$\log \Delta t_i = \log(i(i+1)) + \log\left(\frac{1}{2}P_0\dot{P}\right).$$

The y-intercept would then be $\log\left(\frac{1}{2}P_0\dot{P}\right)$. The gradient of the line is $m = 1$.

The value of $\dot{P}$ is -1.9×10^{-12} seconds per second (this is determined from the graph, see following two pages).

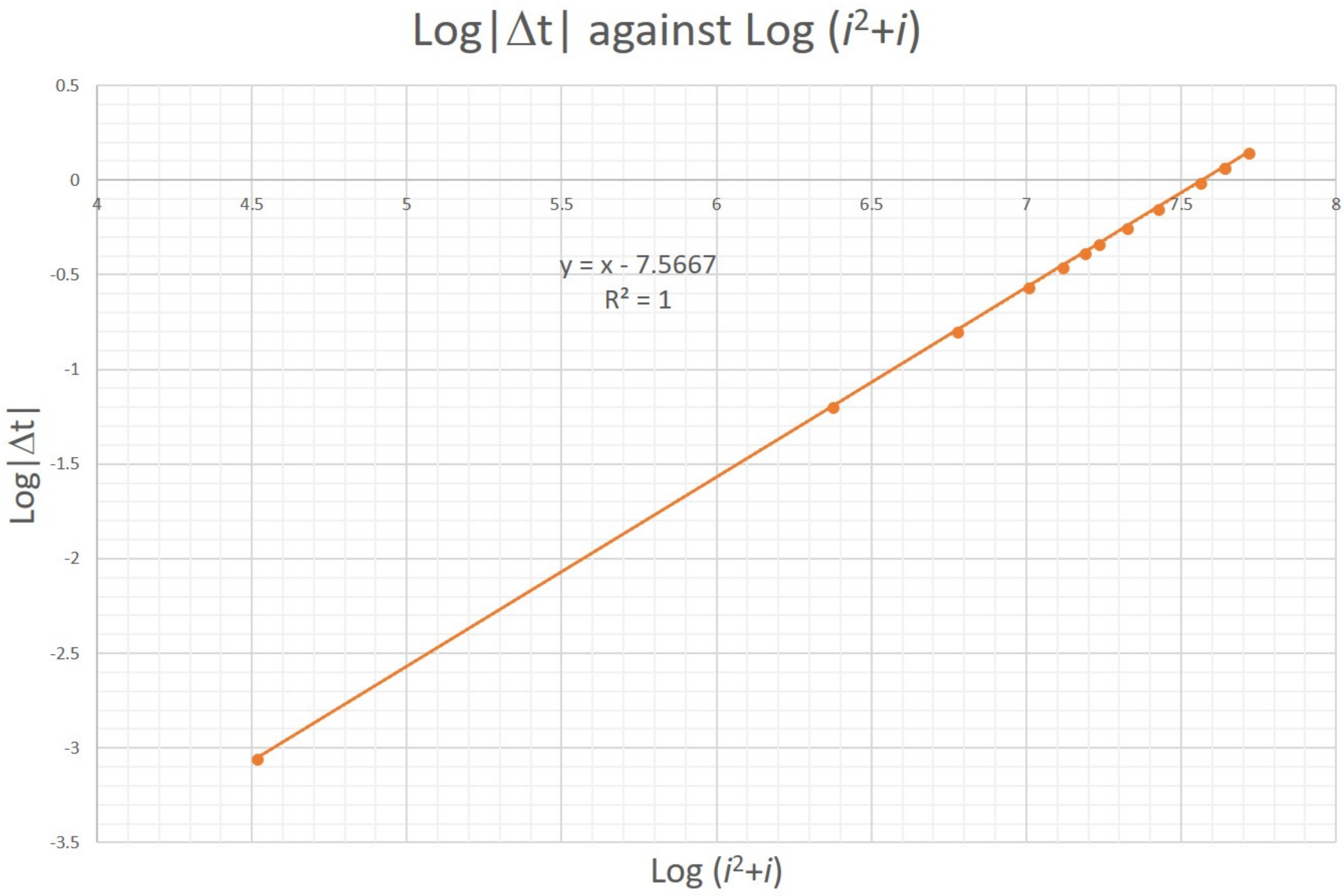
(Additional note: The graph of Δt against cumulative periastron i is as follows.



The interested reader is encouraged to check out this link for more details on this question.)

Approximate Date	Time of Periastron Passage (JED − 2,440,000)	Number of Hours Between 2 Observations	Number of Complete Orbits Between 2 Observations	Actual Cumulative Completed Orbits	Δt_i
1974.77	2331.446	0	0	0	0
1974.93	2389.586	1395.349	180	180	−0.000883681
1976.13	2826.924	10496.126	1354.000036	1534	−0.063867225
1976.93	3118.591	7000.001	903.0000096	2437	−0.161151198
1977.58	3356.640	5713.179	737.0000111	3174	−0.27333482
1977.96	3493.591	3286.822	424.0000088	3598	−0.351226531
1978.23	3593.397	2395.349	309.0000019	3907	−0.414135323
1978.42	3663.488	1682.171	217.0000065	4124	−0.46140991
1978.82	3807.544	3457.365	446.0000028	4570	−0.566593685
1979.31	3988.423	4341.086	560.0000059	5130	−0.713943244
1980.10	4276.537	6914.730	892.0000106	6022	−0.983779849
1980.59	4454.509	4271.319	551.0000043	6573	−1.172027064
1980.59	4455.477	23.256	3.000000635	6576	−1.173097083
1981.14	4656.382	4821.706	622.000003	7198	−1.405491826

Note that the number of complete orbits calculated using the initial period is larger than the actual value, and the actual number of cumulative completed orbit should be rounded down.

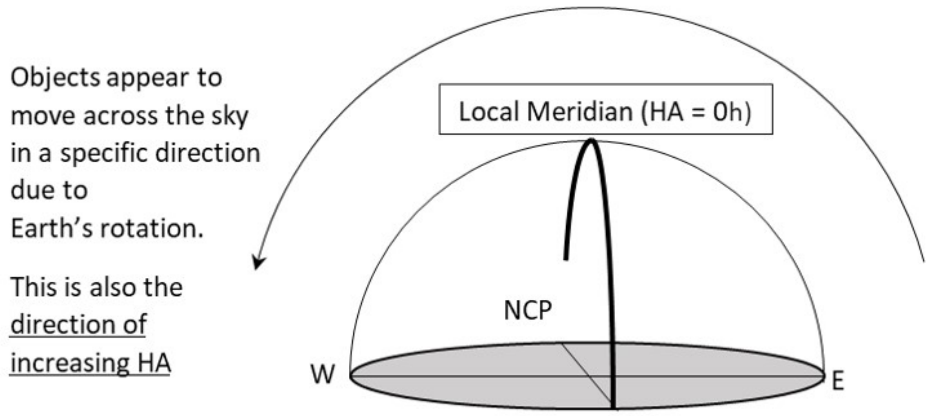

Log|Δt| against Log (i^2+i)
y = x - 7.5667
R² = 1
Log|Δt|
Log (i^2+i)
0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
4
4.5
5
5.5
6
6.5
7
7.5
8

Appendix A

(Adapted from AC 2015 SNR DRQ Q5: Seasons in the Sun)

In astronomy, the concept of the **Hour Angle (HA)** is often used. In general, the HA is the angular distance of a point measured westward from the Local Meridian (the upper arc connecting the zenith to the celestial poles).

The HA can be expressed in terms of angles (degrees/radians/arcseconds etc.), or units of time like hours/minutes/seconds. If we adopt the latter convention, the HA represents the amount of time since the object last crossed the meridian. Indeed, hour angles are often expressed in this form. A diagram for an observer in the Northern Hemisphere is shown below. For clarity, the South Celestial Pole has been omitted.



Determining the HA for any object at a specific time can be computationally demanding. However, calculations for the Sun are greatly simplified if we use solar time, as we can use more natural units. For example, the Sun has an HA of 0 hours at solar noon (as it must be on the Local Meridian then).